

ON THE POSITIVITY OF RIEMANN–STIELTJES INTEGRALS

JANI LUKKARINEN AND MIKKO S. PAKKANEN

ABSTRACT. We study the question, whether a Riemann–Stieltjes integral of a positive continuous function with respect to a non-negative function of bounded variation is positive.

1. INTRODUCTION

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $g : [a, b] \rightarrow \mathbb{R}$ a function of bounded variation. It is a classical result that for such f and g , and for any $y \in (a, b]$, the Riemann–Stieltjes integral

$$\int_a^y f(x) dg(x), \quad (1.1)$$

exists (see, e.g., [3, pp. 316–317]). The purpose of this note is to address the following simple question that seems to be less well-known.

Question 1.2. *If f is positive, g is non-negative, non-vanishing, and satisfies $g(a) = 0$, can we then select the upper limit of integration y so that the integral (1.1) is positive?*

The only reference known to us is a note by Satyanarayana [4] where an affirmative answer is proven under a slightly different set of assumptions, most importantly, assuming that g is *non-decreasing*. Let us stress that *positive* will here always mean *strictly positive*. In particular, since f is continuous, under our assumptions $0 < \min f \leq \max f < \infty$.

Recall that, since g is of bounded variation, there exist non-decreasing functions g^+ and g^- such that $g = g^+ - g^-$. Therefore, finite $\lim_{x \rightarrow y^+} g(x)$ and $\lim_{x \rightarrow y^-} g(x)$ exist for any $y \in [a, b]$ (apart from $\lim_{x \rightarrow a^-} g(x)$ and $\lim_{x \rightarrow b^+} g(x)$, obviously). Moreover, the integral (1.1) is equal to

$$\int_a^y f(x) dg^+(x) - \int_a^y f(x) dg^-(x).$$

Since $g(a) = 0$, we may assume that g^+ and g^- are non-negative and satisfy $g^\pm(a) = 0$.

We may now distinguish two special cases, where it is evident that the answer to Question 1.2 is yes.

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- (1) If $\lim_{x \rightarrow x_L^+} g(x) > 0$, where $x_L := \inf\{x : g(x) > 0\}$, then there exists $\varepsilon > 0$ such that (1.1) is positive for $y = x_L + \varepsilon$.
- (2) If $g^+(y) > 0$ and $g^-(y) = 0$, then (1.1) is at least $g^+(y) \min_{x \in [a, y]} f(x) > 0$.

The first item follows from the elementary lower bound, valid for $0 < \varepsilon \leq b - x_L$,

$$\int_a^{x_L + \varepsilon} f(x) dg(x) \geq g^+(x_L + \varepsilon) \min_{a, x_L - \varepsilon \leq x \leq x_L + \varepsilon} f(x) - g^-(x_L + \varepsilon) \max_{a, x_L - \varepsilon \leq x \leq x_L + \varepsilon} f(x),$$

whereas the second item is a straightforward consequence of $g^\pm(a) = 0$.

In general, the integral (1.1) is positive if and only if

$$\int_a^y f(x) dg^+(x) > \int_a^y f(x) dg^-(x). \quad (1.3)$$

Obviously, the condition $g \geq 0$ is equivalent to $g^+ \geq g^-$, and we have $g(x) > 0$ if and only if $g^+(x) > g^-(x)$. It would be tempting to conjecture that, as a continuous function, f is “nearly constant” in some neighborhood of x_L and, hence, that the inequality (1.3) ought to hold for $y = x_L + \varepsilon$ with some “small” $\varepsilon > 0$, suggesting an affirmative answer to Question 1.2 in general.

Remark 1.4. The proviso $g(a) = 0$ may seem superfluous as the value of the integral (1.1) does not depend on $g(a)$. However, together with the condition $g \geq 0$ it constrains the behavior of g near x_L , which is a key part of the formulation of Question 1.2. It should also be stressed that the continuity of f is equally important—aside from the possible non-existence of the integral, there is no reason to expect the answer to Question 1.2 to be yes when f fails to be continuous.

2. NEGATIVE ANSWER TO QUESTION 1.2

Unfortunately, the heuristic above is too simple-minded since mere continuity does not restrict the *fine properties* of the integrand f and leaves it with enough room to vary “too much” for our purposes. Indeed, the general answer to Question 1.2 is no. We show that for any f that exhibits “enough” variation, there exists a suitable g such that the integral (1.1) is less than zero for all $y \in (a, b]$.

Theorem 2.1. *Let $f : [a, b] \rightarrow (0, \infty)$ be a continuous function. Suppose that there exist two sequences (\underline{x}_n) and (\bar{x}_n) with*

$$a < \cdots < \underline{x}_n < \bar{x}_n < \cdots < \underline{x}_2 < \bar{x}_2 < \underline{x}_1 < \bar{x}_1 \leq b$$

such that for some $\alpha > 0$ and $\gamma \in (0, 1)$,

$$f(\bar{x}_n) - f(\underline{x}_n) \geq \alpha n^{-\gamma} \quad \text{for all } n \in \mathbb{N}.$$

Then, there exists a non-vanishing function $g : [a, b] \rightarrow [0, \infty)$ of bounded variation such that $g(a) = 0$ and

$$\int_a^y f(x) dg(x) < 0 \quad \text{for all } y \in (a, b].$$

Proof. Let $\beta > 1$ and consider $h : [a, b] \rightarrow [0, \infty)$ defined by

$$h(x) := \sum_{n \in \mathbb{N}} n^{-\beta} \chi_{[\underline{x}_n, \bar{x}_n)}(x),$$

where χ_E denotes the characteristic function of a set E . (Figure 1 illustrates the definition for Example 2.2 below.) This is, by construction, a function of bounded variation such that $h(a) = 0$. We have for any $n \in \mathbb{N}$,

$$\int_a^{\underline{x}_n} f(x) dh(x) = n^{-\beta} f(\underline{x}_n) - \sum_{k=n+1}^{\infty} k^{-\beta} (f(\bar{x}_k) - f(\underline{x}_k)),$$

where

$$\begin{aligned} \sum_{k=n+1}^{\infty} k^{-\beta} (f(\bar{x}_k) - f(\underline{x}_k)) &\geq \alpha \sum_{k=n+1}^{\infty} k^{-(\beta+\gamma)} \\ &\geq \alpha \int_{n+2}^{\infty} x^{-(\beta+\gamma)} dx \\ &= \frac{\alpha}{\beta + \gamma - 1} (n+2)^{-(\beta+\gamma-1)}. \end{aligned}$$

Since $\gamma < 1$ and $\sup_{n \in \mathbb{N}} f(\underline{x}_n) < \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_a^{\underline{x}_n} f(x) dh(x) < 0 \quad \text{for all } n \geq n_0.$$

We may also note that for all $n \geq n_0$, with $n \geq 2$, and $y \in (\underline{x}_n, \underline{x}_{n-1})$,

$$\int_a^y f(x) dh(x) \leq \int_a^{\underline{x}_n} f(x) dh(x) < 0.$$

Thus, defining $g := h\chi_{[0, \bar{x}_{n_0})}$ yields a function with all the properties stated in the theorem. \square

Example 2.2. Function $f : [0, 1] \rightarrow (0, \infty)$ given by

$$f(x) := \begin{cases} x^\gamma \sin(1/x) + 2, & \text{if } x > 0, \\ 2, & \text{if } x = 0, \end{cases}$$

where $\gamma \in (0, 1)$, satisfies the condition of Theorem 2.1 with $\bar{x}_n = \frac{1}{4n-3} \frac{2}{\pi}$, $\underline{x}_n = \frac{1}{4n-1} \frac{2}{\pi}$, $n \in \mathbb{N}$, using $\alpha = 2(2\pi)^{-\gamma}$. Figure 1 illustrates the behavior of f and of the corresponding g and Riemann–Stieltjes integral, as defined in the proof of Theorem 2.1.

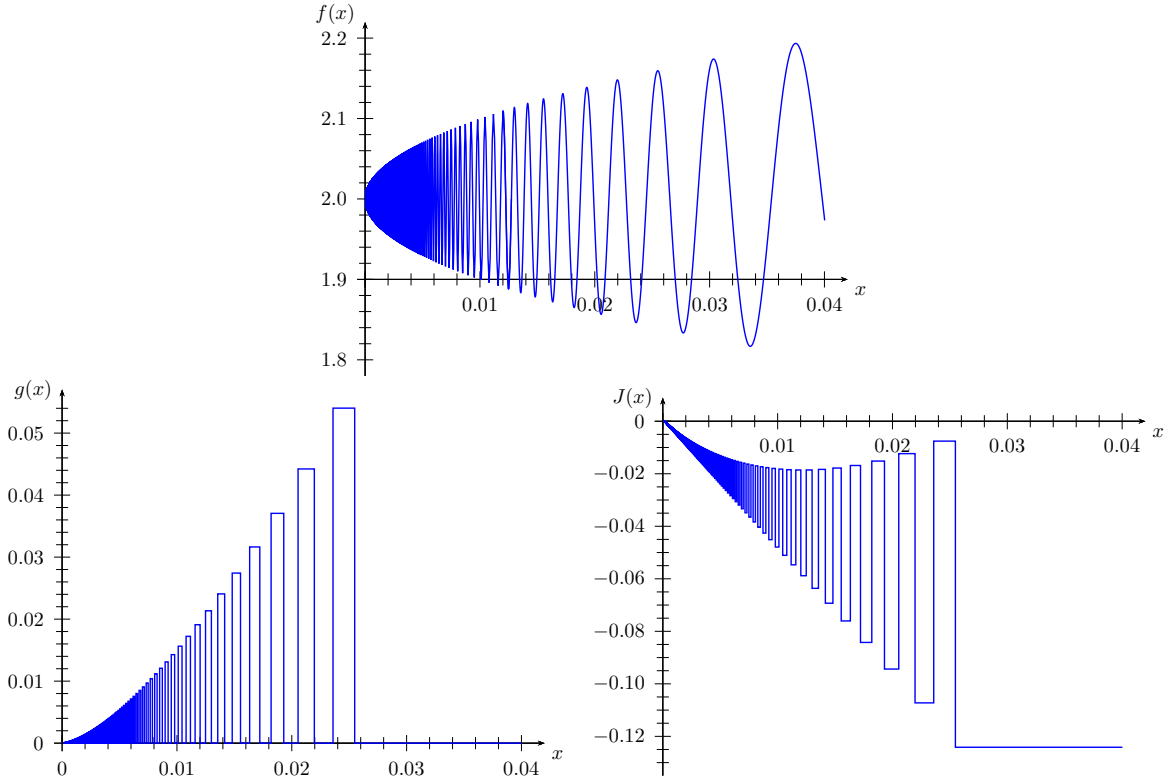


FIGURE 1. Plot of f , as defined in Example 2.2 for $\gamma = \frac{1}{2}$, and the corresponding functions g and J , $J(x) := \int_0^x f(x') dg(x')$, as defined in the proof of Theorem 2.1 using $\beta = \frac{3}{2}$ and $n_0 = 7$. The values have been computed numerically by considering only contributions with $n \leq 1000$. For clarity, only the region with $x \in [0, 0.04]$ is shown.

3. INTEGRANDS OF BOUNDED VARIATION

Any integrand f that satisfies the condition of Theorem 2.1 is clearly of *unbounded variation*. This prompts us to ask, could we actually obtain an affirmative answer to Question 1.2 if f varied “less.” In fact, we are able to show that, if f is of *bounded variation*, the answer to Question 1.2 is yes. Bounds for Riemann–Stieltjes integrals under these assumptions have been derived by Beesak [1], but instead of building our argument on them, we give a direct proof which relies on some elementary measure theory and the measure-theoretic version of *Grönwall’s inequality*.

Theorem 3.1. *Let $f : [a, b] \rightarrow (0, \infty)$ be a continuous function and $g : [a, b] \rightarrow [0, \infty)$ a non-vanishing function of bounded variation such that $g(a) = 0$. If f is of bounded variation, then*

$$\int_a^y f(x) dg(x) > 0 \quad \text{for some } y \in (a, b].$$

Proof. Let us denote by $V_c^d f$ the total variation of f on $[c, d] \subset [a, b]$. Recall that total variation is additive in the sense that $V_a^d f = V_a^c f + V_c^d f$ for $a \leq c \leq d$ [3, Theorem 12.1]. Moreover, since f is continuous, the mapping $x \mapsto V_a^x f$ from $[a, b]$ to $[0, \infty)$ is continuous [3, Theorem 12.2]. Thus, there exists a finite, positive Borel measure ν on $[a, b]$ such that $\nu([c, d]) = V_c^d f$ for any c and d such that $a \leq c < d \leq b$. Let us define another finite, positive Borel measure by $\mu(dx) := f(x)^{-1} \nu(dx)$. By construction, we have then

$$|f(d) - f(c)| \leq \int_{[c,d]} f(x) \mu(dx).$$

By an approximation with suitable Riemann–Stieltjes sums, where the values of g are chosen to be sufficiently close to their respective infima on all subintervals of the partitions, we can prove that

$$\left| \int_a^y g(x) df(x) \right| \leq \int_{[a,y)} f(x) g(x) \mu(dx). \quad (3.2)$$

The Riemann–Stieltjes integral on the left hand side is well-defined by the integration by parts formula [3, Theorem 12.14]

$$\int_a^y g(x) df(x) = f(y)g(y) - f(a)g(a) - \int_a^y f(x) dg(x), \quad (3.3)$$

whenever $\int_a^y f(x) dg(x)$ exists and this is always true under the present assumptions.

Now suppose that, contrary to our assertion, we have

$$\int_a^y f(x) dg(x) \leq 0 \quad \text{for all } y \in (a, b]. \quad (3.4)$$

Rearranging the integration by parts formula (3.3) and using the assumption $g(a) = 0$, (3.2), and (3.4), we obtain for any $y \in (a, b]$

$$f(y)g(y) \leq \int_a^y g(x) df(x) \leq \int_{[a,y)} f(x) g(x) \mu(dx).$$

But the measure-theoretic version of Grönwall’s inequality [2, Theorem A.5.1] implies that then $f(x)g(x) \leq 0$ for all $x \in [a, b]$, whence $g = 0$, a contradiction. \square

Remark 3.5. There are two straightforward refinements to Theorem 3.1. Firstly, it clearly suffices that f is of bounded variation on $[x_L, x_L + \varepsilon]$ for some $\varepsilon > 0$. Secondly, if g is right-continuous, then also the mapping

$$y \mapsto \int_a^y f(x) dg(x)$$

is right-continuous and, under the assumptions of Theorem 3.1, there exists an interval $[c, d] \subset [a, b]$ such that

$$\int_a^y f(x) dg(x) > 0 \quad \text{for all } y \in [c, d].$$

Remark 3.6. Theorem 3.1 explains why the answer to Question 1.2 is perhaps elusive. Experimentation with nicely behaving integrands will not suffice, since a “pathological” f is required in order to discover the general answer. However, such integrands need not be mere curiosities. In fact, our study of Question 1.2 was originally motivated by an application in financial mathematics involving as the integrand a *path* of a continuous-time *stochastic process*, which is typically of unbounded variation.

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JANI LUKKARINEN, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI, P.O. BOX 68, FI-00014 HELSINGIN YLIOPISTO, FINLAND.

E-mail address: `jani.lukkarinen@helsinki.fi`

MIKKO S. PAKKANEN, CREATES AND DEPARTMENT OF ECONOMICS AND BUSINESS, AARHUS UNIVERSITY, BARTHOLINS ALLÉ 10, DK-8000 AARHUS C, DENMARK.

E-mail address: `mss@iki.fi`

URL: <http://www.mikkopakkanen.fi/>